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Radio mean labeling of a graph

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Abstract

A Radio Mean labeling of a connected graph G is a one to one map f from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam}\,(G)$. The radio mean number of f, $rmn(f)$, is the maximum number assigned to any vertex of G.The radio mean number of G, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labelings f of G. In this paper we find the radio mean number of graphs with diameter three, lotus inside a circle, Helms and Sunflower graphs.

Keywords: Radio labeling; Diameter; Wheel; Helms

1. Introduction

Throughout this paper we consider finite, simple, undirected and connected graphs. $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of G. Also, for a graph G, p and q denote the number of vertices and edges respectively. In 2001, Chartrand et al. [1] defined the concept of radio labeling of G. Radio labeling of graphs is motivated by restrictions inherent in assigning channel frequencies for radio transmitters [1]. Radio labeling behavior of several graphs are studied by Kchikech et al. [2,3], Khennoufa et al. [4], Liu et al. [5–9], Van den Heuvel et al. [10] and Zhang [11]. Motivated by the radio labeling we define radio mean labeling of G. A radio mean labeling is a one to one mapping f from $V(G)$ to N satisfying the condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam}\,(G) \tag{1.1}$$

for every $u, v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G. The condition

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(1.1) is called radio mean condition. In this paper we determine the radio mean number of some graphs like graphs with diameter three, lotus inside a circle, gear graph, Helms and Sunflower graphs. Let x be any real number. Then $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms and definitions not defined here are followed from Harary [12] and Gallian [13].

2. Main results

Since any radio mean labeling f is one to one, it follows that $rmn(G) \geq |V(G)|$. Further if $\text{diam}(G) = d$ and $V(G) = \{v_1, v_2, \ldots, v_p\}$, then $f : V(G) \to \mathbb{N}$ defined by $f(v_i) = d + i - 2$, $1 \leq i \leq p$, is a radio mean labeling and hence $rmn(G) \leq p + d - 2$. In particular for any graph with $d = 2$, we have $rmn(G) = p$. Now if G is any graph with diameter 3 and if (u_1, u_2, u_3, u_4) is a diametrical path then f defined by $f(u_1) = 1$, $f(u_4) = 2$, $f(u_3) = 3$ and $f(v)$ for remaining vertices are arbitrarily assigned the labels $4, 5, \ldots, p$, then it can be easily verified that f is a radio mean labeling of G and hence $rmn(G) = p$. Hence the following problem naturally arises:

Problem 2.1. Characterize graphs G for which $rmn(G) = p$.

The following theorem gives another family of graphs G with $rmn(G) = p$.

The sunflower graph SF_n is obtained from a wheel with the central vertex v_0 and the cycle $C_n : v_1 v_2 \ldots v_n v_1$ and additional vertices $w_1 w_2 \ldots w_n$ where w_i is joined by edges to v_i, v_{i+1} where v_{i+1} is taken modulo n.

Theorem 2.1. *The radio mean number of the sunflower graph SF_n is its order.*

Proof. For $n \leq 5$, since $\text{diam}(SF_3) = 2$ and $\text{diam}(SF_4) = \text{diam}(SF_5) = 3$, the result follows. Assume $n \geq 6$. It is clear that $\text{diam}(SF_n) = 4$. Define the function f with co domain $\{1, 2, \ldots, 2n + 1\}$ as follows: $f(w_1) = 1$, $f(w_2) = n$, $f(w_3) = 2$, $f(w_i) = i - 1$, $4 \leq i \leq n$, $f(v_0) = n + 1$ and $f(v_i) = n + 1 + i$, $1 \leq i \leq n$. We must show that the radio mean condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 5 \tag{2.1}$$

for every pair of vertices (u, v) where $u \neq v$.

Now, if either $f(u) \geq 6$ or $f(v) \geq 6$, then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 4$ and hence (2.1) trivially holds. Hence let $1 \leq f(u)$, $f(v) \leq 5$. Clearly $u, v \in \{w_1, w_3, w_4, w_5, w_6\}$. If $u = w_i$ and $v = w_j$ and $|i - j| > 1$, then $d(u, v) = 3$ or 4 and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 2$. Therefore (2.1) holds. If $u = w_i$, $v = w_{i+1}$, then $d(u, v) = 2$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 3$. Hence (2.1) holds. $\square$

The Helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle C_n.

Theorem 2.2. *The radio mean number of a Helm H_n is $2n + 1$.*

Proof. Let $W_n = C_n + K_1$ where C_n is the cycle $u_1 u_2 \ldots u_n u_1$ and $V(K_1) = \{u_0\}$. Let w_i be the pendent vertex adjacent to u_i $(1 \leq i \leq n)$. Since $\text{diam}(H_3) = 3$, the result follows. Now let $n \geq 4$. Then $\text{diam}(H_n) = 4$. We define f on V as follows: $f(w_i) = i$ for all i with $1 \leq i \leq n$ and $f(u_i) = n + 1 + i$ for all i with $0 \leq i \leq n$.

Since $\text{diam}(H_n) = 4$, to prove that f is a radio mean labeling, we need to prove that

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 5 \tag{2.2}$$

for every pair of vertices (u, v) where $u \neq v$.

If either $f(u) \geq 6$ or $f(v) \geq 6$, then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 4$ and hence (2.2) trivially holds. Hence let $1 \leq f(u), f(v) \leq 5$. If $n \geq 5$, it follows that $u, v \in \{w_1, w_2, w_3, w_4, w_5\}$. If $u = w_i$, $v = w_j$ and $|i - j| > 1$, then $d(u, v) = 4$ and (2.2) holds. If $u = w_i$ and $v = w_{i+1}$, then $d(u, v) = 3$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 2$. Hence (2.2) holds. Thus f is a radio mean labeling of H_n. If $n = 4$, then $u, v \in \{w_1, w_2, w_3, w_4, u_0\}$ and since $f(u_0) = 5$, the inequality (2.2) holds. $\square$

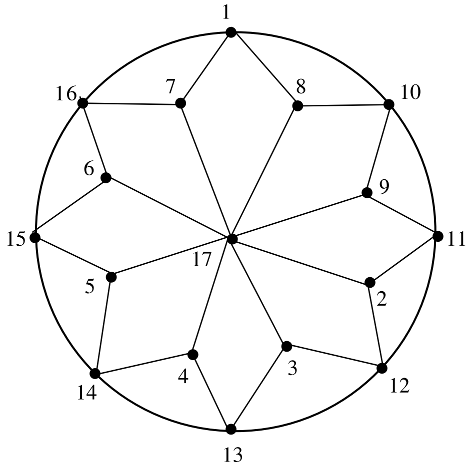

Fig. 1.

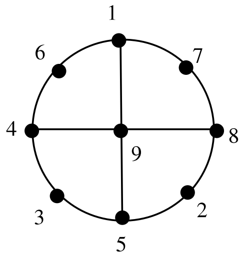

Fig. 2.

The graph lotus inside a circle LC_n is obtained from the cycle $C_n : w_1w_2 \ldots w_nw_1$ and a star $K_{1,n}$ with central vertex u and the end vertices $u_1u_2 \ldots u_n$ by joining each u_i to w_i and $w_{i+1 (\text{mod } n)}$.

Theorem 2.3. $rmn\,(LC_n) = 2n + 1$.

Proof. Suppose $3 \le n \le 7$, then since $\operatorname{diam}(LC_n) = \begin{cases} 2 & \text{if} \quad n = 3, 4 \\ 3 & \text{if} \quad n = 5, 6, 7 \end{cases}$, the result follows. Let $n \ge 8$. Here $\operatorname{diam}(LC_n) = 4$. We define an injective map $f : V(LC_n) \to \{1, 2, \ldots, 2n + 1\}$ by $f(u_i) = i + 1$, $1 \le i \le n$, $f(w_{n-1}) = 1$, $f(w_n) = n + 2$, $f(w_i) = n + 2 + i$, $1 \le i \le n - 2$, $f(u) = 2n + 1$. Now it is enough to prove that

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \ge 5 \tag{2.3}$$

for every pair of vertices (u, v) where $u \ne v$.

If either $f(u) \ge 6$ or $f(v) \ge 6$, then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 4$ and hence (2.3) holds.

Let $1 \le f(u), f(v) \le 5$. If $n \ge 5$, it follows that $u, v \in \{w_{n-1}, u_1, u_2, u_3, u_4\}$. Clearly $d(w_{n-1}, u_i) = 3$ where $i \in \{1, 2, 3, 4\}$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 2$. Then (2.3) holds. Also $d(u_i, u_j) = 2$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 3$. Therefore (2.3) holds. Hence $rmn\,(LC_n) = 2n + 1$. $\square$

Illustration 2.1. *A radio mean labeling of LC_8 with radio mean number 17 is given in Fig. 1.*

The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.

Theorem 2.4. *The radio mean number of a gear graph G_n is $2n + 1$.*

Proof. Let $W_n = C_n + K_1$ where C_n is the cycle $v_1 v_2 \dots v_n v_1$ and $V(K_1) = \{v\}$. Let $V(G_n) = V(W_n) \cup \{u_i : 1 \le i \le n\}$ and $E(G_n) = E(W_n) \cup \{v_i u_i, u_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_n u_n, u_n v_1\} - E(C_n)$. Since $\operatorname{diam}(G_3) = 3$, the result follows. For $n = 4$, the result follows from Fig. 2.

Assume $n \ge 5$. In this case, the diameter of G_n is 4. First consider the vertex u_1 and assign the label 1 to it. Then move two steps on the rim in the clockwise direction and reach the vertex v_3. Assign the label 2 to v_3. Then move one step and reach the vertex v_4. Let it be labeled by 3. The remaining successive vertices of the rim are labeled $4, 5, 6, \dots$ respectively. Note that v_1 receives the label $2n-3$. Finally assign the labels $2n-2$, $2n-1$, $2n$ and $2n+1$ respectively to the vertices v_2, u_2, u_3 and v.

Now we must check the radio mean condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \ge 5 \tag{2.4}$$

for every pair of vertices (u, v) where $u \ne v$.

If either $f(u) \ge 6$ or $f(v) \ge 6$, then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 4$ and hence (2.4) holds.

Let $1 \le f(u), f(v) \le 5$. If $n \ge 5$, it follows that $u, v \in S = \{w_{n-1}, u_1, u_2, u_3, u_4\}$. Let $u = u_1$ and v be any other vertex in S, then $d(u, v) = 3$ or 4. In this case $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 2$. Hence (2.4) holds. If $u, v \in \{v_4, u_4, v_5\}$ then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 4$ and hence the inequality (2.4) holds. Finally, consider $u = v_3$ and $v \in \{u_4, v_4, v_5\}$. Here $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 3$ and $d(u, v) = 2$ or 3. Thus (2.4) holds. □

Theorem 2.5. *Let $G_{n,m}$ be a graph obtained by identifying any two vertices of the wheels W_n and W_m. Then*

$$rmn(G_{n,m}) = \begin{cases} 10 & \text{if } m = n = 4 \text{ and any two rim vertices are identified} \\ m + n + 1 & \text{Otherwise.} \end{cases}$$

Proof. Call the rim vertices of W_n by u_i $(1 \le i \le n)$ and that of W_m by v_j $(1 \le j \le m)$ and u, v denote the central vertices of the respective wheels.

Case 1. Identify a rim vertex of one wheel and the central vertex of the other.

In this case $\operatorname{diam}\left(G_{n,m}\right) = 3$ and therefore the result follows.

Case 2. Identify the two central vertices.

Here, $\operatorname{diam}\left(G_{n,m}\right) = 2$. Therefore, the result is true.

Case 3. Any two rim vertices are identified.

In this case if $m = n = 3$ then $\operatorname{diam}(G_{n,m}) = 2$ and if $m = 3$ or $n = 3$ then the graph is of diameter 3. In both the cases, the result is obviously true. Now we assume that $m \ge 4$ and $n \ge 4$. In this case $\operatorname{diam}(G) = 4$. Without loss of generality assume $n \le m$.

Subcase 1. $m = 4$.

Here $n = 4$. Here $n = 4$. In this case Fig. 3, shows that the vertex labels satisfy the radio mean condition. Therefore

$$rmn\,(G) \le 10. \tag{2.5}$$

Claim. $rmn(G_{n,m}) > 9$.

Let $u_1 u_2 u_3 u_4 u_1$ and $v_1 v_2 v_3 u_4 v_1$ be the cycles of the two wheels. Take $u_4 = v_1$. Clearly 1 and 2 should be labeled to the vertices with a distance atleast 3. If $f(u_1) = 1$ then $f(v_3) = 2$. In this case 3 should not be the label of any vertices. If $f(u_2) = 1$ then $f(v_2) = 2$. This forces $f(v_3) = 3$. In this case 4 should not be a label of any vertices. Hence

$$rmn(G_{n,m}) > 9. \tag{2.6}$$

From Eqs. (2.5) and (2.6) the result follows.

Subcase 2. $m \ge 5$.

Let $u_r = v_s$ be the identified vertex. Assign the label 1 to the vertex u_j with $j \notin \{r+1, r, r-1\}$. Then assign the label 2 to the vertex v_{s+1}. Now we move to the vertex v_{s+3}. Let it be labeled by 3. After that assign the labels

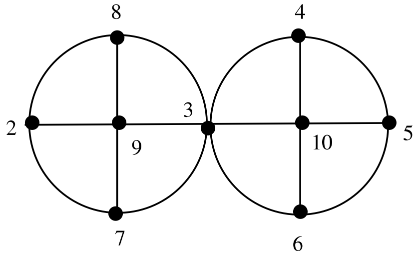

Fig. 3.

$4, 5, \ldots$ consecutively to the rim vertices. Note that the last vertex in this sequence, that is, v_s, receives the label m. Then assign the next label $m+1$ to the vertex v_{s+2}. Now we move to the other wheel W_n. Label all the rim vertices of W_n except u_j and u_r (already u_j and u_r receive the labels 1 and m respectively) by the next available labels from $\{m+2, m+3, \ldots, m+n-1\}$ in any order. Finally assign the labels $m+n$ and $m+n+1$ to the central vertices of W_n and W_m respectively.

Now we must check the radio mean condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 5 \tag{2.7}$$

for every pair of vertices (u, v) where $u \neq v$.

If either $f(u) \geq 6$ or $f(v) \geq 6$, then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 4$ and hence (2.7) holds. Consider the case $1 \leq f(u), f(v) \leq 5$. If $m \geq 6$, then $u, v \in S = \{u_j, v_{s+1}, v_{s+3}, v_{s+4}, v_{s+5}\}$. Suppose $u = u_j$ and v is any other vertex of S then $d(u, v) = 3$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 2$. Hence (2.7) holds. If $u = v_{s+1}$ and $v \in \{v_{s+3}, v_{s+4}, v_{s+5}\}$ then $d(u, v) = 2$ and $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 3$. Hence (2.7) holds. If $u, v \in \{v_{s+3}, v_{s+4}, v_{s+5}\}$ then $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 4$. Therefore (2.7) holds. In the case of $m = 5$, $f(v_s) = 5$ and $u, v \in \{u_j, v_s, v_{s+1}, v_{s+3}, v_{s+4}\}$. Here, it is easy to verify the inequality (2.7).

Hence $rmn\left(G_{n,m}\right) = m + n + 1$. □

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